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Algebra in Ancient India

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1. INTRODUCTION

Algebra is the sub-topic of mathematics in which letters and symbols are used to represent the numbers. Various Ancient Indian Mathematicians like Brahmgupta, Bhaskara-II and Aryabhatta developed various algebraic concepts and have worked extensively in the development of the Algebraic Sciences. Majority of the manuscripts related to the mathematics in ancient India discovered have still not been explored but the small part that has been understood finds extensive usage and applications in the algebra which is being studied in the present times. The methods include addition, subtraction, multiplication and division of algebraic expressions along broader applications such as factorization of the algebraic expressions, solutions of various equations of one or multiple degrees. In addition to the basic operations, the concepts of Ancient Indian Algebra also have the application and usage in the factorization of algebraic expressions and in linear, quadratic, cubic and bi-quadratic equations in one and multiple variables.

As we move further, we would like to present a shloka from the book 'Bhrahmsphuta-siddhanta' (Chapter-9; Sutra 44) authored by the Great Indian Mathematician Brahmgupta, to give a glimpse of the glorious past of Indian Tradition:

वर्गाचतुर्गुणितां रूपाणं मध्यवर्गसहितानाम।
मूलं मध्येनोनं वर्गद्धिगुणोदवृतं मध्य।। [9, Sutra 44]

Through this Shloka**,** Brahmgupta has tried to present the solution of the quadratic equation:

The above Shloka, means multiply four to the absolute quantity and coefficient of the square of unknown quantity add square of middle coefficient and then decrease by middle coefficient, divide by twice of the coefficient of unknown quantity. This paper is divided into two sections: the first describes the fundamental operations on algebraic expressions and the second one showcasing the factorization techniques and methods of solving the linear, quadratic, cubic and biquadratic equations.

2. FUNDAMENTAL OPERATIONS IN ALGEBRAIC EXPRESSIONS

Though there have been advancements in the studies of Ancient Indian Mathematics. Many scholars and researchers have worked upon books by Ancient Indian Mathematicians like Brahmsphutasiddhanta, Lilavati, Bijaganita etc. to name a few. In the present paper, the methods and formulae described are based on the collection of 13 Sutras (सूत्र Formulae) and 16 Sub-sutras (उप-सूत्र- Sub-Formulae) which find a direct or indirect application in the domain of Algebraic Studies.Through this section, we will describe various methods with their meaning and applications on Fundamental Operations:

- ❖ Addition of Algebraic expressions
- ❖ Subtraction of Algebraic expressions
- ❖ Multiplication of Algebraic expressions
- ❖ Division of Algebraic expressions

2.1. Addition of Algebraic Expressions

The addition of algebraic expressions can be described as under:

(i) Creation of Columns with variables at the top.

(ii) Then, coefficients of the similar variables are written as per their respective columns.

(iii) Then, we add the coefficients in respective elements.

(iv) Finally, the added coefficients are written along with respective variables to form the resultant expression.

It can be understood with the help of the below example: To add 3x-4y+4z , 2x+3y-5z and 4z-2x-2y:

X	y	z
3	-4	4
2	3	-5
-2	-2	4
3	-3	3

Hence, the added expression can be written as:

3x-3y+4z

Similarly, we can add three or more algebraic expressions.

2.2. Subtraction of Algebraic Expressions

The subtraction of algebraic expressions can be described as under:

(i) Creation of columns with variables at the top.

(ii) Coefficients of the similar variables are written as per their respective columns.

(iii) Then, we add the coefficients in respective elements. (iv) Finally, the added coefficients are written along with respective variables to form the resultant expression.

It can be understood with the help of the below example:

To subtract 2x+3y+5z from the expression 3x-4y+4z, below is the actual worked out example:

As we see, we are subtracting the coefficients of the respective variables arranged in a column.

x	y	z
3	-4	4
2	3	5
(-)	(-)	(-)
1	-7	-1

Hence, the resultant expression can be written as: x-7y-z

The same method can be used for subtracting expressions with higher number of variables and degree too.

2.3. Multiplication of Algebraic Expressions

The multiplication of Algebraic Expressions can be done in multiple ways. The methods within the scope of this paper are 1) By Sutra Nikhilam and 2) By Sutra Urdhva Triyak. They have been broadly discussed in the following part:

2.3.1. By Sutra Nikhilam (Completion of Base)

This sutra takes any one variable as a base and is based out on deviation of the expression with the base. Its working method for application can be understood as under**:**

a) This method involves using one term of the algebraic expression as Base and the rest part as deviation (difference from Base).

b) In this method, we multiply the chosen Base with the Sum of One expression and deviation of the other on the left side and the multiple of the deviations on the right side then both are added to get the result.

It can be understood with the help of the example below: To multiply (x+4) and (x+5)

x+4	4
x+5	5
x(x+4+5)	20

Hence, the answer is $x^2 + 9x + 20$. Here, we chose *x* as base and hence +4 and +5 are deviations from the base.

2.3.2. By Sutra Urdhva Triyak (Straight and Criss- Cross Multiplication)

This method is based on criss-cross and straight multiplication of the algebraic terms. The process for its application can be understood as under:

a) In this method, the terms on the extreme right and extreme left are multiplied to each other.

b) Rest terms are multiplied leaving one each from the left side moving towards right with criss-cross where even number of terms are involved and straight for the middle one with criss-cross for the rest if terms are odd.

It can be understood with the following example:

To multiply (r+4) (r+5):

r r 4 + 4

r r 5 +5

$= r^2 + (5r + 4r) + 20$

Hence, the result is $r^2 + 9r + 20$.

2.4. Division of Algebraic Expressions

By Sutra Paravartya Yojayet (परावर्त्ययोजयेत्): As the name of the sutra says with its literal meaning as 'Transpose and Adjust'. In this method, following steps are involved:

a) We write the divisor at the left most part, the dividend in such a manner that the left most, then the dividend in such a manner that the left most

b) Or the third part contain the same number of expressions which are in the divisor other than the highest degree of variable say (n – 1) expressions where n is the number of expressions in the divisor. It can be understood with the following example:

$(8x^2 - 6x + 4 + x + 1)/(x+1)$

Divisor $\underline{x + 1}$ $8x^2$-6 $\underline{4}$ Dividend -14

	8 -6	
-8		$\underline{14}$

Quotient 8 -14 18 Remainder

Directions:

a) We note down the coefficient of x^2 at quotient place just below the first expression.

b) Then we multiply 8x-1 = -8, which we write below -6 w2hich is coefficient of x. Adding -6 and -8 we write the sum at quotient place.

c) Again we again we multiply -14 and -1 to get 14 which we write below 4and get remainder 18.

d) Hence, the quotient is 8x – 14 and the remainder is 18.

3. BROADER ALGEBRAIC CONCEPTS BY ANCIENT INDIAN MATHEMATICAL METHODS-I

This section of the paper discusses the broader concepts of algebra which are listed as under:

❖ Factorization of Algebraic Expressions

❖ Finding the Highest Common Factor of Algebraic Expressions

3.1. Factorization of Algebraic Expressions

There are multiple methods of finding the factors of Algebraic Expressions in the Vedic Mathematics. We shall be discussing the below methods:

1. By Sutra Vilokanam (meaning Observation).
2. By Reverse of Sutra Nikhilam
3. By Sutra Sankalana-Vyavakalanabhyam
4. By Sutra Anurupyena
5. Remainder Theorem

3.1.1. By Sutra Vilokanam (meaning Observation)

This method is used to find the factors of the algebraic expressions of degree one. The following is the working method for better understanding:

a. Like expressions are grouped and then common elements are factored out.

b. Commonly factored elements are again factored out. This can be understood with the following example:

Factorizing the expression: $2x+2y+zx+zy$

$=2x+2y+zx+zy$

$=2(x+y)+z(x+y)$

$=(x+y)(2+z)$

Hence, the result is $(x+y)(2+z)$.

3.1.2. By Reverse of *Sutra Nikhilam*

Reverse of *Sutra Nikhilam* can be applied to find out the Factors of Quadratic Equations. The following are the steps involved:

1. Constant term = Product of both Deviations; The Product Pair is chosen such that their sum is equal to the coefficient of the middle term.
2. We rewrite the factors in the Nikhilam Form.
3. Finally, we rewrite the factors.

This can be understood with the following example: Find the factors of $x^2- 3x – 40$.

$x^2-3x-40$

$=x(x-3)-40$

$=x(x-8+5)|(-8)x5$

$=x-8|-8$

$=x+5|5$

$=(x-8)(x+5)$

Directions:-

a. Constant term = Product of both deviation

b. Possible factor (-8 , 5) , (-20 , 2) , (-5 , 8) , (2 , -20) , (4 , -10) , (10 , -4).

c. Since, sum of deviations = -3. Hence, we choose (-8 , 5) d. We write the expression in Nikhilam form.

e. Hence $(x – 8)(x + 5)$ are the factors.

3.1.3. Sutra Sankalan-Vyavakalan

This method is based on addition and subtraction of Deviations as *Sankalan* means joining whereas *Vyavakalan* means breaking. When in a quadratic algebraic expression the middle term is zero, then this method is very suitable. Its steps are as follows:

a. Square Root of the first Expression is chosen as base. b. The second term is the product of both deviations.

c. Rest method is the same as Nikhilam Sutra Reverse. This can be understood well with the following example:- Finding the factors of the expression:$36x^2$-$25y^2$

=6x(6x) | 5y(-5y)

=6x (6x + 5y – 5y) | (5y) (-5y)

=6x+5y|5y

6x+5y|-(5y)

=(6x +5y) (6x – 5y)

Directions:

a. Product of both deviations = $-25y^2$
b. Base = 6x
c. Middle term = 0. Hence, we put +5y and -5y as deviations. d. We put 6x with 5y and – 5y.
e. Write the expression in reverse for to get the factor.

3.1.4. By Sutra Anurupyena (meaning 'Same Ratio')

This formula is based on the similar ratios of the split and then grouped expressions. In this method of finding the factors, we break the expression in the following form:

'First Term of the expression: First Part of the Mid-term' equals the ratio' Second Term of the expression: Second Part of the Mid-term'

This can be understood with the following example:

$18x^2 + 34x^2+16$

$=18x^2 + 18x + 16x + 16$

=18x (x + 1) + 16 (x + 1)

=(18x + 16) (x + 1)

Here, we see the below relationship while we break the middle term: $18x^2{}_{:\,18x}$ = x : 1

16x : 16 = x : 1

3.1.5. Remainder Theorem

The remainder theorem states that Dividend (Expression) equals the product of Quotient and Divisor added with the remainder. Similarly, if the remainder is zero then the Dividend is equal to the product of Quotient and Divisor. Thus, for if (x-a) and the divisor are the factors of a given expression then x=a must bring the result as zero as per the Remainder Theorem.

To understand this let us find one factor of $x^2 + 5x + 6$. Putting x=(-2), we get:

$x^2 + 5x + 6$

$=(-2)^2+5(-2)+6$

=0

Hence, (x+2) is a factor of $x^2 + 5x + 6$.

3.2. Finding the Highest Common Factor

To find the Highest Common Factor '*Sutra Lopana Sthapanabhyam*' is used. In this, method the highest and the lowest powers of the expression are removed using methods of Addition, Subtraction, Multiplication and Division as needed.

This can be understood with the following example:

To find the Highest Common Factor of the expressions: $x^2 + 7x+6$ and $x^2 -5x-6$

We first add the expressions:

$x^2 + 7x+6$

$\underline{x^2 -5x-6}$

$2x^2+2x$

Then we subtract the expressions:

$x^2 + 7x+6$

$\underline{(-)x^2 -5x-6}$

$12x+12$

Factorizing *Using Sutra Vilokanam*, we separate out the common factors in both results and hence is the HCF

$=2x(x+ 1)$

$=12(x+1)$

Since, (x+1) is the common factor in both expressions. Hence, x+1 is the HCF.

4. BROADER ALGEBRAIC CONCEPTS BY ANCIENT INDIAN MATHEMATICAL METHODS-II

This section of the paper discusses the broader concepts of algebra which are listed as under:

- ❖ Understanding Equations
- ❖ Linear Equations in One Variable
- ❖ Quadratic Equations in One variable
- ❖ Cubic/Biquadratic Equations in One Variable

4.1. Understanding Equations

Equations are formed when two different Algebraic Expressions are equated. For example: 2x+5=3x 7

Solution (Roots) of an equation: The values of the variables for which both sides of the equations become equal, are referred to as the solution of the equations.

Types of the Equations: Depending upon the number of variables and their highest power (degree) the equations are termed as linear (degree1), quadratic (degree2), cubic (degree3) and so on, in one/two or respective number of variables.

4.2. Linear Equations in One variable

The equations which contain only one degree variable is termed as Linear Equation and if it contains only one variable in it, then it is termed as Linear Equations in One Variable. There are many ways of solving the linear equations in one variable. The linear equations in one variable can be solved by application of the '***Sutra Paravartya Yojayet***' meaning 'Transpose and Adjust'.

4.2.1. For Equations of Type. ax+b=cx+d: The value of x is equal to:

$$x=\frac{d-b}{a-c}$$

Stated with an example: Solve the equation 3x+7=2x+9.

$$x=\frac{9-7}{3-2}$$

Or x=2

4.2.2. For the linear equations of the type: (x+a)(x+b)=(x+c)(x+d); The solution is given as:

$$x=\frac{cd-ab}{a+b-c-d}$$

It can be visualized with the following example:

(x+3)(x+4)=(x-5)(x-6)

$$x=\frac{30-12}{3+4-(-5)-(-6)}$$

Or x= 18/18

Or x=1

4.2.3. Similarly, for the linear equations of the type **(x+a)(x+b)=(x+c)(x+d).**

The solution is given as:

$$x=\frac{cd-ab}{a+b-c-d}$$

It can be visualized with the following example: (x+3)(x+4)=(x-5)(x-6).

$$x=\frac{30-12}{3+4-(-5)-(-6)}$$

Or x= 18/18

Or x=1

4.2.4. For the linear equations of the type:

$$\frac{ax+b}{cx+d}=\frac{p}{q}$$

The value of x can be found as:

$$x=\frac{pd-bq}{aq-cp}$$

Stated with an example:

$$\frac{3x+3}{4x+4}=\frac{-6}{4}$$

Thus the value of

$$x=\frac{(4)(-6)-(3)(4)}{(4)(4)-5(-6)}$$

Or x= 18/7

4.2.5. For the Linear Equations of the type

$$\frac{m}{x+a}+\frac{n}{x+b}=0$$

The solution can be stated as:

$$x=\frac{mb-na}{m+n}$$

It can be understood with the following example:

$$\frac{2}{x+3}+\frac{3}{x+4}=0$$

Thus, the value of x can be found as:

$$x=\frac{(2)(4)-(3)(3)}{2+3}$$

Or $x= -1/5$

4.3. Solution of the linear equations using '*Sunyam Samyasamuccaye*' Method (शून्यं_साम्यसमुच्चये:)

It means when the '*Samuccaya*' is the same the '*Samuccaya*' is zero. Here, *Samuccaya* has different meanings as under:

4.3.1. A term occurring as a common factor in both sides of the equation

Stated with an example: $13x+3x=5x+3x$. Since, x is the common term hence, $x=0$.

4.3.2. Product of the independent terms

$(x+9)(x+8)=(x-12)(x-6)$. Since, $(9)(8)=(-12)(-6)$. Hence, $x=0$.

4.3.3. Sum of denominators of two fractions

$[2/(x+3)] + [3/(x+4)] = 0$

Thus, $x+3+x+4=0$ or $x=7/2$.

4.3.4. The sum of Numerators and Denominators if are equal or equivalent (with a numeric factor) then, any one of them equated to 0 gives us the solution

Stated with an example: $(3x+4)/(6x+7)=(x+1)/(2x+3)$

Thus, $2(n1+n2) = d1+d2$;

$(n1+n2) = 4x +5=0$.

Or $x=-5/4$

Here, n=numerator; d= denominators

4.4. Quadratic Equations in One Variable

Equations with highest power two in one variable are termed as Quadratic Equations in two variables. There are many methods of solving the Quadratic Equation. We will be discussing Sub-sutra 'Vilokanam' (Observation ववलोकनम्सूत्र), 'Sunyam- Samyasamuccaye' (शून्यं_साम्यसमुच्चये:), and the Sunyam Anyat Method(शून्यमन्यत्) and Mix of Sunyam Anyat and Paravartya (Merger) method.

4.4.1. By Sutra Vilokanam (ववलोकनम्सूत्र) or the Reciprocal Method

The literal meaning of the 'Vilokanam' is to view or to observe. This method is suitable for the equations of the type: x+ (1/x) = n.

For example: x+ (1/x) =10/3 Or x+ (1/x) = 3+1/3
Thus, by observation we have x=3, 1/3.

4.4.2. Using Sutra 'Sunyam- Samyasamuccaye' (शून्यं_साम्यसमुच्चये) Method

This method is suitable for the equations of the type: (ax+b)/(cx+d)=(ex+f)/(gx+h). Here, if the sum of the numerators and denominators of both sides are equal and also the difference of the numerator and denominators of the same side are also equal on both sides then both are equated to zero to find the roots.

For example: (7x+5)/(9x-5)=(9x+7)/(7x+17)
Thus, adding N1+N2=16x+12=D1+D2=0 and also,
N1-D1=2x-10=N2-D2=0.
Thus, we have x=5 or x= (-3/4).
Here, N=Numerator; D=Denominator

4.4.3. Sunyam Anyat Method. This method is used for the equations of the type [a/(x+a)]+[b/(x+b)]=[c/(x+c)]/[d/(x+d)].

For this method to be used the following conditions must be fulfilled:

a. a + b=c + d
b. (a/a)+(b/b)=(c/c)+(d/d)

Then one of the roots is zero and the other can be found out by adding the denominators D1+D2 and equating them to 0.

To understand with an example:

[5/(x+5)]+[3/(x+3]=[6/(x+6)]+[2/(x+2)]

Since, 5+3=6+2; also (5/5)+(3/3)=(6/6)+(2/2).

Thus, one root is 0. To find the other one, we add the denominators; x+5+x+3=0 or 2x+8=0 or x=-4.

4.5. Cubic/Bi-quadratic Equations in One Variable

The Cubic/Quadratic Equations are the equations with degree three/four also have many ways of solving the equation. We will have one of them the 'Purana- Apurnabhyam' (पुरणपुराणभ्यं) which means the completion of the cubic/biquadratic.

4.5.1. Solving a cubic equation

Cubic Equations can be solved by means of completing the cubic for an incomplete cubic.
It can be understood with an example:
$x^3-6x^2+11x-6=0$
Therefore, $x^3-6x^2=-11x+6$
Using the value of $(x-2)^3=x^3-6x^2+12x-8$
Thus, $x^3-6x^2+12x+8=-11x+6+12x-8$
Or $(x-2)^3=(x-2)$
Or $(x-2)=0$ and $(x-2)=\pm 1$
Or x=1,2,3

4.5.2. Solving a biquadratic equation

Bi-quadratic Equations can be solved by means of completing the biquadratic for an incomplete bi quadratic.
It can be understood with an example:
$r^4-12r^3+49r^2-78r+40=0$
Therefore, $r^4 - 12r^3 = -49r^2+78r-40$…(i)
Now, since $(r-3)^4= r^4 - 12r^3+54r^2-108r+81$…(ii)
Or $(r-3)^4=5r^2-30r-41$
Or $(r-3)^4=(r-3)(5x-15)-4$
Or $(r-3)^4-5(r-3)^2+4=0$
Or $(r-3)^4-4(r-3)^2-1(r-3)^2+4=0$
Or $(r-3)^2[(r-3)^2-4]-1[(r-3)^2-4]=0$
Or $[(r-3)^2-4][(r-3)^2-1]=0$
Or $[(r-3)^2-4]=0$ and $[(r-3)^2-1]=0$
Or $(r-3)=\pm2$ and $(r-3)=\pm1$
Or r=4,5,2 and 1

5. CONCLUSION

The paper includes various concepts used in the ancient Indian Times in the sub-domain algebra of the mathematics discipline. Through this paper, we strive to bring the scholars the practical insights on the usage and applications of the concepts of algebra at the primary, middle and secondary schooling level. This paper will also serve as guide to the students at the schooling level to encourage the curiosity for studying and exploring the Vedic Mathematics from various available resources. This paper is an effort to show the students, the fact that the problems in algebra can even be solved even mentally and just by observation.

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